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Dirac Wave Equation in the de Sitter Universe

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We present and discuss the Dirac wave equation in the de Sitter universe. This equation is obtained by factoring the second-order Casimir invariant operator associated to the Fantappié–de Sitter group.

1. INTRODUCTION

The de Sitter space is the curved space-time which has been most studied by quantum field theorists because it and the anti-de Sitter space are the unique maximally symmetric curved spacetimes (Weinberg, 1972).

Some papers published many years ago discussed the Dirac wave equation in the de Sitter universe. Hara *et al.* (1954) and Ueno (1953, 1954) studied the unified description of elementary particles, and Takeno (1952, 1953) the spherically symmetric problem in general relativity. Ikeda (1953) gave a five-dimensional representation of the electromagnetic and electron field equations in a curved space-time, which was then compared with the formalism proposed by Dirac in the case of the de Sitter space-times. Goto (1954) discussed an equation of Dirac–Fierz type in the de Sitter universe. More recently Gürsey (1962) presented an introduction to the de Sitter group, with a discussion of the structure of the group, commutation relations, invariants, and the generators of the de Sitter group, which are rotation operators in a five-dimensional Euclidean space. Gürsey (1963) presented the Casimir operators for the de Sitter group and concluded by showing that a particle in a de Sitter universe does not have a definite mass and spin, but definite eigenvalues of the two Casimir invariant operators of the group. These authors

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did not discuss the equations obtained from the invariant Casimir operators. This theme was the subject of studies by Börner and Dürr (1969), who discussed the de Sitter space-times and derived an eigenvalue equation for the second-order Casimir invariant operator using the representations for this group.

Now, an original way to study the cosmological problem is through the theory of hyperspherical models of the universe proposed by Fantappié (1973) and developed by Arcidiacono (1976, 1977, 1978, 1986, 1995). In this theory it is necessary to distinguish absolute space-time (with constant curvature), the effective seat of physical events, from the infinite relative space-times (tangents) where each observer localizes and sees the phenomena. Then we use a flat representation of the de Sitter universe on one of the tangent spaces. Among the infinite representations we use the Beltrami (1865) geodesic representation where the geodesics of the hyperspherical space-time correspond to the straight lines of the flat tangent space-time of the observer's localization. It follows that the group of motions in itself of the de Sitter universe is represented by the so-called Fantappié–de Sitter group, isomorphic to the five-dimensional pseudo-rotation group, i.e., by projectives which change the Cayley–Klein absolute (Arcidiacono, 1986, 1995).

Recently, Arcidiacono and Capelas de Oliveira (1991a) discussed the Laplace equation and d'Alembert wave equation (Arcidiacono and Capelas de Oliveira, 1991b,c) in the de Sitter universe, using the techniques proposed by Fantappié and Arcidiacono, in terms of ultraspherical polynomials. More recently, Capelas de Oliveira (1992) discussed the homogeneous d'Alembert generalized wave equation for the case of a physical situation involving a small distance (a local problem) using the same technique. In another recent paper (Notte Cuello and Capelas de Oliveira, 1995) we proposed a new construction of the Casimir invariant operators for the Fantappié–de Sitter group using the same techniques mentioned above. In that paper we obtained the commutation relations and Casimir invariant operators for the Fantappié–de Sitter group and in a second paper (Capelas de Oliveira and Notte Cuello, 1996) we discussed the Klein–Gordon wave equation. This equation has been obtained from the second-order Casimir invariant operator. In the present paper we obtain the Dirac wave equation in the de Sitter universe and discuss its solutions. This equation is obtained by factoring the second-order Casimir invariant operator into two linear factors in the angular moment operators. Using original equations for a system of *four* partial differential equations and solving this system, we obtain solutions in terms of spherical harmonics with spin-weight.

This paper is organized as follows: in Section 2 we present some basic concepts of spinors, spin-weights, and the spherical harmonic spinors; in Section 3 we present a brief review of the technique proposed by Fantappié and Arcidiacono; in Section 4 we briefly discuss the Fantappié–de Sitter group and the Casimir invariant operators; in Section 5 we obtain the Dirac wave equation and in Section 6 we discuss and solve the Dirac equation in a limiting case.

2. SPIN-WEIGHTS

It is known that a Pauli spinor ψ can be represented geometrically by a flag (Torres Del Castillo, 1991; Maiorino *et al.*, 1993a,b), the pole of which is the vector (real) $R = (R_1, R_2, R_3)$ given by

$$R_i = \psi^\dagger \sigma_i \psi \tag{2.1}$$

where † denote adjoint. The direction of the flag is given by the direction of the real part of the vector $M = (M_1, M_2, M_3)$ defined by

$$M_i = \psi^T \epsilon \sigma_i \psi \tag{2.2}$$

where T denotes transposition, the σ_i are the Pauli matrices, and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.3}$$

The components of a spinor can be parametrized by

$$\psi = \sqrt{r} e^{-ix/2} \begin{pmatrix} \cos\theta/2 \; e^{-i\phi/2} \\ \sin\theta/2 \; e^{i\phi/2} \end{pmatrix} \tag{2.4}$$

although at $\theta = 0$ and $\theta = \pi$ the parameters x and ϕ are not well defined. Using equations (2.1) and (2.2), we get

$$\begin{aligned} R &= (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r\cos\theta) = r\hat{e}_r \\ M &= re^{-ix}(\hat{e}_\theta + i\hat{e}_\phi) \\ &= r[(\cos x\hat{e}_\theta + \sin x\hat{e}_\phi) + i(-\sin x\hat{e}_\phi + \cos x\hat{e}_\phi)] \end{aligned} \tag{2.5}$$

where $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ are orthogonal vectors tangent to the spherical coordinates lines.

For the purpose of studying functions defined on the sphere of unitary radius, we introduce the spinor

$$\vartheta \equiv \begin{pmatrix} \cos\theta/2 \; e^{-i\phi/2} \\ \sin\theta/2 \; e^{i\phi/2} \end{pmatrix} \tag{2.6}$$

which is obtained by setting $r = 1$ and $x = 0$ in equation (2.4), and another spinor defined by

$$l \equiv \epsilon\overline{\vartheta} = \begin{pmatrix} \sin\theta/2 \; e^{-i\phi/2} \\ -\cos\theta/2 \; e^{i\phi/2} \end{pmatrix} \tag{2.7}$$

The above spinors satisfy the relation

$$l^A\vartheta_A = l^A\epsilon_{AB}\vartheta^B = l^1\vartheta^2 - l^2\vartheta^1 = 1 \tag{2.8}$$

at each point of the sphere; ϑ and l form a basis for spinor space and induce a basis for vectors and the tensors of arbitrary rank (Torres Del Castillo, 1991; Maiorino *et al.*, 1993a,b).

A quantity η has spin-weight s if under the transformation

$$\vartheta' = e^{i\alpha/2}\vartheta \tag{2.9}$$

it transforms according to

$$\eta' = e^{is\alpha}\eta \tag{2.10}$$

Thus, the spinor 0 has spin-weight 1/2 and $l' = \epsilon\overline{0}' = \epsilon e^{-i/2}\overline{0} = e^{-i\alpha/2}l$; l has spin-weight $-1/2$. Also, if η has spin-weight s, then $\overline{\eta}$ has spin-weight $-s$. Another way to say that is: η has spin-weight s if under a rotation of the orthonormal basis $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ (generated from the spherical coordinates) by an angle α about $\hat{e}_r$, it transforms according to equation (2.10). Since under this rotation $\hat{e}_r$ is invariant, $\hat{e}_r$ has spin-weight zero, but $\hat{e}_\theta$ and $\hat{e}_\phi$ transform according to $\cos\alpha\hat{e}_\theta - \sin\alpha\hat{e}_\phi$ and $\sin\alpha\hat{e}_\theta + \cos\alpha\hat{e}_\phi$, respectively and thus we cannot define the spin for $\hat{e}_\phi$ and $\hat{e}_\phi$, i.e., "they have no spin-weight." But the combination $\hat{e}_\theta \pm \hat{e}_\phi$ transforms according to

$$(\cos\alpha\hat{e}_\theta - \sin\alpha\hat{e}_\phi) \pm i(\sin\alpha\hat{e}_\theta + \cos\alpha\hat{e}_\phi) = e^{\pm i\alpha}(\hat{e}_\theta \pm i\hat{e}_\phi)$$

i.e., $(\hat{e}_\theta + i\hat{e}_\phi)$ and $(\hat{e}_\theta - i\hat{e}_\phi)$ have spin-weight one and minus one, respectively.

Any vector field F in R^3 can be written as

$$F = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_\phi\hat{e}_\phi \tag{2.11}$$

with the components F_r, F_θ, and F_ϕ determined from $F_r = F\cdot\hat{e}_r$, $F_\theta = F\cdot\hat{e}_\theta$, and $F_\phi = F\cdot\hat{e}_\phi$, and given that $\hat{e}_r$ and $\hat{e}_\theta \pm \hat{e}_\phi$ have spin-weight zero, one, and minus one, respectively, the combinations of the components of F, $F_r = F\cdot\hat{e}_r$, and $F_\theta \pm iF_\phi = F\cdot(\hat{e}_\theta \pm i\hat{e}_\phi)$ have spin-weight zero, one, and minus one, respectively.

Introducing

$$F_+ \equiv F_\theta + iF_\phi \qquad \text{and} \qquad F_- \equiv F_\theta - iF_\phi \tag{2.12}$$

in equation (2.11), we obtain

$$F = F_r\hat{e}_r + \tfrac{1}{2}F_-(\hat{e}_\theta + i\hat{e}_\phi) + \tfrac{1}{2}F_+(\hat{e}_\theta - i\hat{e}_\phi) \tag{2.13}$$

which expresses an arbitrary field vector F in terms of components that have defined spin-weight.

Now, we can introduce the operators $\eth$ and $\bar{\eth}$ (Newman and Penrose, 1966) that act on quantities η of spin-weight s according to the following expressions:

$$\eth\eta \equiv -\sin^s\theta\left(\frac{\partial}{\partial\theta} + i\,\frac{\partial}{\partial\phi}\right)(\sin^{-s}\theta)\eta \tag{2.14}$$

$$\bar{\eth}\eta \equiv -\sin^{-s}\theta\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\,\frac{\partial}{\partial\phi}\right)(\sin^{s}\theta)\eta$$

where the quantities $\eth\eta$ and $\bar{\eth}\eta$ have spin-weight $s + 1$ and $s - 1$, respectively.

Now using the identity $\nabla \times (\nabla \times F) = \nabla(\nabla\cdot F) - \nabla^2F$ and that $\eth\bar{\eth} = \bar{\eth}\eth$ when acting on quantities of spin-weight equals to zero (in general, if η has spin-weight s, $\bar{\eth}\eth\eta - \eth\bar{\eth}\eta = 2s\eta$) (Newman and Penrose, 1966), we obtain

$$\begin{aligned}\nabla^2F = &\left[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r^2}\bar{\eth}\eth F_r + \frac{1}{r^2}\eth F_- + \frac{1}{r^2}\bar{\eth}F_+\right]\hat{e}_r\\ &+\left[\frac{1}{2r}\frac{\partial^2}{\partial r^2}(rF_-) + \frac{1}{2r^2}\bar{\eth}\eth F_- - \frac{1}{r^2}\bar{\eth}F_f\right](\hat{e}_\theta + i\hat{e}_\phi)\\ &+\left[\frac{1}{2r}\frac{\partial^2}{\partial r^2}(rF_+) + \frac{1}{2r^2}\eth\bar{\eth}F_+ - \frac{1}{r^2}\eth F_r\right](\hat{e}_\theta - i\hat{e}_\phi)\end{aligned} \tag{2.15}$$

Then we can obtain an additional simplification of the Laplacian of a vector field, using the spherical harmonic with spin-weight (${}_sY_{lm}$), which for integer values of s are functions given by (Newman and Penrose, 1966),

$${}_sY_{lm} = \begin{cases}\left[\dfrac{(l-s)!}{(l+s)!}\right]^{1/2}\eth^sY_{lm}, & 0 \le s \le l\\ (-1)^s\left[\dfrac{(l+s)!}{(l-s)!}\right]^{1/2}\bar{\eth}^{-s}Y_{lm}, & -l \le s \le 0\\ 0, & |s| > l\end{cases} \tag{2.16}$$

where Y_{lm} denote the usual spherical harmonics, which are functions with spin-weight equal to zero. Since the operators $\eth$ and $\bar{\eth}$ change the spin-weight by one and minus one, respectively, from the definition (2.16), we see that ${}_sY_{lm}$ has spin-weight s (${}_0Y_{lm} \equiv Y_{lm}$).

The definition (2.16) is equivalent to the relations

$$\eth({}_sY_{lm}) = [(l - s)(l + s + 1)]^{1/2}{}_{s+1}Y_{lm} \tag{2.17}$$

$$\bar{\eth}({}_sY_{lm}) = -[(l + s)(l - s + 1)]^{1/2}{}_{s-1}Y_{lm}$$

and then we have

$$\bar{\eth}\eth({}_sY_{lm}) = -(l - s)(l + s + 1){}_sY_{lm} \tag{2.18}$$
$$\eth\bar{\eth}({}_sY_{lm}) = -(l + s)(l - s + 1){}_sY_{lm}$$

The numerical factors in (2.17) and (2.18) are normalization factors such that, for a fixed value of s, the spherical harmonic with spin-weight ${}_sY_{lm}$ (where $l = |s|, |s| + 1, \ldots, m = -l, -l + 1, \ldots, l$) define an orthogonal set

$$\int_0^{2\pi}\int_0^{\pi} {}_s\bar{Y}_{lm}(\theta, \phi)\, {}_sY_{l'm'}(\theta, \phi) \sin\theta\, \phi\theta\, \phi\theta = \delta_{ll'}\delta_{mm'} \tag{2.19}$$

We note that we use the usual spherical harmonic Y_{lm} that appears in equation (2.16) normalized. Also, the set of the ${}_sY_{lm}$, with s fixed, is a complete set in the sense that any function $f(\theta, \phi)$ with spin-weight s can be expressed as a linear combination as follows:

$$f = \sum_{l=|s|}^{\infty} \sum_{m=-l}^{l} a_{lm}({}_sY_{lm}) \tag{2.20}$$

where the coefficients a_{lm} are constants given by

$$a_{lm} = \int_0^{2\pi}\int_0^{\pi} \overline{{}_sY_{lm}(\theta, \phi)} f(\theta, \phi) \sin\theta\, d\theta\, d\phi \tag{2.21}$$

3. A BRIEF REVIEW OF THE METHOD

In this section we present a brief review of the method proposed by Fantappié and Arcidiacono. Consider a five-dimensional space E_5 with the homogeneous coordinates and the four-dimensional Beltrami coordinates in the de Sitter space.

Therefore the five-dimensional homogeneous coordinates denoted by ξ_A ($A = 0, 1, 2, 3, 4$) and the four-dimensional coordinates denoted by χ_μ ($\mu = 0, 1, 2, 3$) are related by

$$\chi_\mu = R\frac{\xi_\mu}{\xi_4} \tag{3.1}$$

satisfying the relation of normalization $\xi_A\xi_A = R^2$, where R is the radius of the de Sitter universe.

Introducing the following notation for the Cayley–Klein absolute,

$$A^2 = 1 + \alpha^2 - \gamma^2 = 1 + \alpha_\mu\alpha_\mu \tag{3.2}$$

where

$$\alpha_\mu = \frac{1}{R}\chi_\mu \qquad \text{and} \qquad \gamma = \frac{1}{R}ct \tag{3.3}$$

we can remove the ξ_4 coordinate and obtain

$$\xi_4 = \frac{R}{A} \qquad \text{and} \qquad \xi_\mu = \frac{\chi_\mu}{A} \tag{3.4}$$

where A is given by (3.2).

To obtain the relation for the partial derivatives, we consider a function $\varphi(\xi_A)$, a homogeneous function of degree N in all five variable ξ_A, and use Euler's theorem for homogeneous functions; we have

$$\xi_A \partial_A \varphi(\xi_A) = N\varphi(\xi_A) \tag{3.5}$$

where we have put $\partial_A \equiv \partial/\partial\xi_A$.

Using the definition of homogeneous functions we can write

$$R^N \varphi(\xi_A) = (\xi_A)^N \varphi(R, \chi_\mu) \tag{3.6}$$

where the function on the right side of the above equation is a function obtained from $\varphi(\xi_A)$ with $\xi_4 \to R$ and $\xi_\mu \to \chi_\mu$.

Taking the derivative of the above equation, first in relation to ξ_4 and second in relation to ξ_μ, and introducing a function $\psi(\chi_\mu)$ by

$$\psi(\chi_\mu) = A^{-N}\varphi(R, \chi_\mu) \tag{3.7}$$

we obtain, respectively,

$$R\frac{\partial}{\partial\xi_4}\varphi(\xi_A) = \left(\frac{N}{A} - A\chi_\mu \frac{\partial}{\partial\chi_\mu}\right)\psi(\chi_\mu) \tag{3.8a}$$

and

$$\frac{\partial}{\partial\xi_\mu}\varphi(\xi_A) = \left(A\frac{\partial}{\partial\chi_\mu} + \frac{N}{AR^2}\chi_\mu\right)\psi(\chi_\mu) \tag{3.8b}$$

which is the link between the two formulations. Then we have solved the problem of passing from the five-dimensional formulation ξ_A to the spacetime formulation χ_μ, i.e., in orthogonal Cartesian coordinates.

4. CASIMIR INVARIANT OPERATORS

In this section we summarize the properties of the Fantappié–de Sitter group, giving a suitable representation of the respective generators and the Casimir invariant operators.

The Fantappié–de Sitter group, isomorphic to the five-dimensional pseudo-rotation group, is the group of motions admitted by a cosmological space with line elements given by

$$-ds^2 = A^2 d\chi_\mu d\chi_\mu = A^2[(d\chi_1)^2 + (d\chi_2)^2 + (d\chi_3)^2 + (d\chi_0)^2] \quad (4.1)$$

where $\chi_0 = ict$ and $R^2A^2 = R^2 + \rho^2 + \chi_0^2$ and $\rho^2 = (\chi_1)^2 + (\chi_2)^2 + (\chi_3)^2$. This space can be embedded in a flat five-dimensional spacetime, with the χ_μ the Beltrami projection from the "sphere" with equation

$$\sum_{A=0}^{4} \xi_A\xi_A = (\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 + (\xi_4)^2 - (\xi_0)^2 = R^2$$

The coordinates are related by equation (3.4) and the differential operators by equations (3.8a) and (3.8b).

The representation of the generators of the Fantappié–de Sitter group (Notte Cuello and Capelas de Oliveira, 1995) is given by the five-dimensional angular momentum operators

$$J_{AB} = -i\hbar\left(\xi_A \frac{\partial}{\partial\xi_B} - \xi_B \frac{\partial}{\partial\xi_A}\right) \equiv L_{AB} \quad (4.2)$$

where A, $B = 0, 1, 2, 3, 4$. In terms of the Beltrami coordinates these are given by

$$L_{\mu\nu} = \chi_\mu p_\nu - \chi_\nu p_\mu \quad (4.3a)$$

and

$$\pi_\lambda \equiv \frac{1}{R} L_{4\lambda} = A^2 p_\lambda + \frac{1}{R^2} \chi_\mu L_{\lambda\mu} \quad (4.3b)$$

where $\nu, \mu, \lambda = 0, 1, 2, 3$.

We note that in the above equations (where π_μ are the analogues of the momentum operators which represent the generators of the Poincaré group acting as transformation group in Minkowski space-time) the linear momentum p_μ and the angular momentum $L_{\mu\nu}$ mix in a unique tensor. This mixing is due to the fact that transformations of the displacement are the analogues of the translation and therefore the energy and the momentum operators ($p_\mu \to -i\hbar\partial_\mu$) are not conserved in relation to the Fantappié–de Sitter group; however, the quantities (4.3b) are conserved.

Introducing the spherical coordinates and defining the operators

T_0 = temporal translations, with $T_0 \equiv \left(\frac{-ic}{R}\right) L_{40}$

T_μ = spatial translations, with $T_\mu \equiv \left(\frac{1}{R}\right) L_{\mu 4}$

V_μ = center-of-mass inertial momentum, with $V_\mu \equiv \left(\frac{-i}{c}\right) L_{0\mu}$

L_μ = spatial rotations, with $L_\mu \equiv L_{\nu\lambda}$

where μ, ν, λ = 1, 2, 3, we obtain the two invariant operators of the Fantappié–de Sitter group (Casimir operators) using T_0, T_μ, V_μ, and L_μ:

$$\mathcal{I}_2 = -\left(T^2 - \frac{1}{c^2} T_0^2\right) - \frac{1}{R^2}(L^2 - c^2V^2) = M^2 \tag{4.4a}$$

and

$$\mathcal{I}_4 = -(\mathbf{L}\cdot\mathbf{T})^2 + \frac{1}{c^2}(T_0\mathbf{L} + c^2\mathbf{T}\times\mathbf{V})^2 + \frac{c^2}{R^2}(\mathbf{L}\cdot\mathbf{V})^2 = N^2 \tag{4.4b}$$

where M^2 and N^2 are constants.

We note that in the limit $R \to \infty$ we obtain

$$\mathcal{I}_2 \to m^2 \qquad \text{and} \qquad \mathcal{I}_4 \to m^2 s(s+1)$$

where m and s are, respectively, the rest mass and the spin which characterize the representations of the Poincaré group (Gürsey, 1963). Then the representations of the Fantappié–de Sitter group are labeled by eigenvalues of $\mathcal{I}_2$ and $\mathcal{I}_4$ which generalize the usual mass and spin. Yet a particle in a Fantappié–de Sitter universe does not have well-defined mass and spin, but only eigenvalues of the $\mathcal{I}_2$ and $\mathcal{I}_4$ invariant operators.

5. FACTORIZATION OF THE SECOND-ORDER CASIMIR INVARIANT OPERATOR

We can write the second-order Casimir invariant operator (4.4a) for the Fantappié–de Sitter group using the ten operators L_{ab} as

$$\tfrac{1}{2}L_{ab}L_{ab} = -R^2M^2 \tag{5.1}$$

where $L_{ab} = -L_{ba}$ and $a, b = 0, 1, 2, 3, 4$, where $M^2 = m^2 + 3i\hbar m/R$.

This formula is discussed in Goto (1954), where m is a real scalar. We see that for $R \to \infty$ we have $M^2 \to m^2$.

We recall from (4.2) that

$$L_{ab} = \xi_a p_b - \xi_b p_a = -i\hbar\left[\xi_a \frac{\partial}{\partial \xi_b} - \xi_b \frac{\partial}{\partial \xi_a}\right]$$

Then we can write

$$\tfrac{1}{2}L_{ab}L_{ab} = (\xi\xi)(pp) - (\xi P)^2 + 3ih(\xi P) \tag{5.2}$$

where

$$\xi\xi = \xi_a\xi_a; \qquad pp = p_a p_a; \qquad \xi = (\xi_0, \ldots, \xi_4); \qquad p = (p_0, \ldots, p_4)$$

Substituting (5.2) in (5.1) with M^2 given above, we get

$$(\xi\xi)(pp) - (\xi p)^2 + 3i\hbar(\xi p) + R^2m^2 + 3i\hbar mR = 0 \tag{5.3}$$

Using the identity

$$(\tfrac{1}{2}\gamma_a\gamma_b L_{ab})^2 = -(\xi\xi)(pp) - 3i\hbar(\gamma\xi)(\gamma p) + (\xi p)^2 \tag{5.4}$$

where $\gamma = (\gamma_0, \ldots, \gamma_4)$ and the γ_a are 4×4 matrices that satisfy

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{ab}$$

noting that

$$\gamma_a\gamma_b L_{ab} = 2(\gamma\xi)(\gamma p) - 2(\xi p) \tag{5.5}$$

and then substituting (5.4) and (5.5) in (5.3), we obtain

$$(\tfrac{1}{2}\gamma_a\gamma_b L_{ab})^2 - R^2m^2 + 3\hbar i(\tfrac{1}{2}\gamma_a\gamma_b L_{ab} - mR) = 0 \tag{5.6}$$

We can write two factors linear in the L_{ab} operators,

$$(\tfrac{1}{2}\gamma_a\gamma_b L_{ab} - Rm)(\tfrac{1}{2}\gamma_a\gamma_b L_{ab} + Rm + 3\hbar i) = 0 \tag{5.7}$$

or

$$\tfrac{1}{2}\gamma_a\gamma_b L_{ab} - Rm = 0 \tag{5.8a}$$

$$\tfrac{1}{2}\gamma_a\gamma_b L_{ab} + Rm + 3\hbar i = 0 \tag{5.8b}$$

where (5.8a) is the same as postulated by Dirac (1935).

Before we discuss the solutions of equation (5.8a) using the method proposed by Fantappié and Arcidiacono [equation (5.8b) is totally analogous], we quickly show that it reduces to the usual Dirac equation in the limit $R \to \infty$. We have

$$\lim_{R\to\infty} \gamma_a\gamma_\lambda\pi_\lambda = -i\gamma'_\lambda p_\lambda$$

where π_λ is given in (4.3b) and the four matrices $\gamma'_\lambda = i\gamma_0\gamma_\lambda$ are Hermitian and satisfy the same commutation rules as the four γ_λ (Gürsey, 1962). Thus, in the limit we obtain

$$\gamma'_\lambda p_\lambda\psi = im\psi \tag{5.9}$$

where m^2 is the limit of M^2 for $R \to \infty$. This shows that equation (5.8a) gives Dirac's equation in flat space if M^2 has a real limit m^2.

Now, we can write equation (5.8a) in terms of the Beltrami coordinates introducing the operators given above:

$$\left\{\gamma_0\gamma_1 icV_1 + \gamma_0\gamma_2 icV_2 + \gamma_0\gamma_3 icV_3 - \gamma_0\gamma_4 \frac{i}{c} RT_0 + \gamma_1\gamma_2 L_3 - \gamma_1\gamma_3 L_2 \right.$$

$$\left. + \gamma_1\gamma_4 RT_1 + \gamma_2\gamma_3 L_1 + \gamma_2\gamma_4 RT_2 + \gamma_3\gamma_4 RT_3 - Rm\right\} = 0 \qquad (5.10)$$

We obtain the explicit form for equation (5.10) in relativistic spherical coordinates (t, r, θ, ϕ):

$$\begin{bmatrix} R\left(1 + \dfrac{t^2}{R^2}\right)I_2 & -r\left(1 - \dfrac{t}{R}\right)\sigma_r \\ -r\left(1 + \dfrac{t}{R}\right)\sigma_r & -R\left(1 + \dfrac{t^2}{R^2}\right)I_2 \end{bmatrix} \frac{\partial\psi}{\partial t}$$

$$+ \begin{bmatrix} \dfrac{rt}{R} I_2 & \left(t + \dfrac{r^2 + R^2}{R}\right)\sigma_r \\ \left(t - \dfrac{r^2 + R^2}{R}\right)\sigma_r & \dfrac{-rt}{R} I_2 \end{bmatrix} \frac{\partial\psi}{\partial r}$$

$$+ \begin{bmatrix} -i\sigma_\phi & \dfrac{t + R}{r}\sigma_\theta \\ \dfrac{t - R}{r}\sigma_\theta & -i\sigma_\phi \end{bmatrix} \frac{\partial\psi}{\partial\theta}$$

$$+ \begin{bmatrix} ir\sigma_\theta & (t + R)\sigma_\phi \\ (t - R)\sigma_\phi & ir\sigma_\theta \end{bmatrix} \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} - \frac{Rm}{i\hbar} I_4\psi = 0 \qquad (5.11)$$

where $\sigma_\theta \equiv \sigma\cdot\hat{e}_\theta$, $\sigma_\phi \equiv \sigma\cdot\hat{e}_\phi$, $\sigma_r \equiv \sigma\cdot\hat{e}_r$ and I_2, I_4 are the 2×2 and 4×4 identity matrices, respectively, and we use for the γ_a matrices

$$\gamma_\lambda = \gamma_0\alpha_\lambda \qquad \text{and} \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$$

with

$$\alpha_\lambda = \begin{bmatrix} 0 & \sigma_\lambda \\ \sigma_\lambda & 0 \end{bmatrix}, \qquad \gamma_0 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}, \qquad \lambda = 1, 2, 3$$

We put $\psi = [{u \atop v}]$, where u and v are two-component spinors, and write $u = u_-\vartheta - v_+l$, $v = v_-\vartheta_- - v_+l$, where the spinors ϑ and l are introduced in Section 2; we use the identities

$$\sigma\cdot\nabla u = \sigma_r\frac{\partial u}{\partial r} + \sigma_\phi\frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} + \sigma_\theta\frac{1}{r}\frac{\partial u}{\partial\theta}$$

$$\sigma\cdot\nabla(u_-\vartheta) = \left(\frac{1}{r}\frac{\partial}{\partial r}(ru_-)\right)\vartheta + \left(\frac{1}{r}\,\eth u_-\right)l \tag{5.12}$$

$$\sigma\cdot\nabla(u_+l) = \left(\frac{1}{r}\,\eth u_+\right)\vartheta - \left(\frac{1}{r}\frac{\partial}{\partial r}(ru_+)\right)l$$

We also note from Section 2 that the following identities hold:

$$\frac{\partial\vartheta}{\partial\phi} = \frac{i}{2}\,\delta\vartheta; \qquad \frac{\partial\vartheta}{\partial\theta} = \frac{-1}{2}\,l; \qquad \frac{\partial l}{\partial\theta} = \frac{1}{2}\,\vartheta; \qquad \frac{\partial l}{\partial\phi} = \frac{i}{2}\,\delta l$$

$$\sigma_\theta\vartheta = -l; \qquad \sigma_\phi\vartheta = -il; \qquad \sigma_r\vartheta = \vartheta; \qquad \sigma_\theta l = -\vartheta; \qquad \sigma_\phi l = i\vartheta$$

$$\sigma_r l = -l; \qquad \sigma_\theta\delta\vartheta = \sin\theta\vartheta + \cos\theta l; \qquad \sigma_\theta\delta l = \sin\theta l - \cos\theta\vartheta \tag{5.13}$$

where

$$\delta \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then we can write equation (5.11), using the identities (5.12) and (5.13) and using the fact that the set $\{\vartheta, -l\}$ is linearly independent, as a system of four partial differential equations:

$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial u_-}{\partial t} - \left(1 - \frac{t}{R}\right)\frac{\partial v_-}{\partial t} + (t + R)\frac{\partial v_-}{\partial r} - (t + R)\frac{1}{r}\,\bar{\eth} v_+$$

$$+ \frac{r^2}{R}\frac{\partial v_-}{\partial r} + \frac{rt}{R}\frac{\partial u_-}{\partial r} + \bar{\eth} u_+ - (1 + k)u_- + (t + R)\frac{1}{r}\,v_- = 0$$

$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_+}{\partial r} + r\left(1 - \frac{t}{R}\right)\frac{\partial v_+}{\partial t} + (t + R)\frac{v_+}{\partial r} - (t + R)\frac{1}{r}\eth v_-$$

$$- \frac{r^2}{R}\frac{\partial v_+}{\partial r} + \frac{rt}{R}\frac{\partial u_+}{\partial r} - \eth u_- - (1 + k)u_+(t + R)\frac{1}{r}v_+ = 0 \tag{5.14}$$

$$r\left(1 + \frac{t}{R}\right)\frac{\partial u_-}{\partial t} + R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_-}{\partial t} - (t - R)\frac{\partial u_-}{\partial r} + (t - R)\frac{1}{r}\bar{\eth}u_+$$

$$+ \frac{r^2}{R}\frac{\partial u_-}{\partial r} + \frac{rt}{R}\frac{\partial v_-}{\partial r} - \bar{\eth}v_+ + (1 + k)v_- - (t - R)\frac{1}{r}u_- = 0$$

$$r\left(1 + \frac{t}{R}\right)\frac{\partial u_+}{\partial t} - R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial v_+}{\partial t} - (t - R)\frac{\partial u_+}{\partial r} - (t - R)\frac{1}{r}\eth u_-$$

$$+ \frac{r^2}{R}\frac{\partial u_+}{\partial r} - \frac{rt}{R}\frac{\partial v_+}{\partial r} - \eth v_- - (1 + k)v_+ - (t - R)\frac{1}{r}u_+ = 0$$

where $k = Rm/i\hbar$.

Now, equation (5.14) can be solved using the method of separation of variables. Using the fact that the set of spherical harmonics with spin-weight is complete, we look for a solution of the form

$$\begin{aligned} u_- &\equiv g(r, t)_{-1/2}Y_{jm}(\theta, \phi) \\ u_+ &\equiv G(r, t)_{1/2}Y_{jm}(\theta, \phi) \\ v_- &\equiv f(r, t)_{-1/2}Y_{jm}(\theta, \phi) \\ v_+ &\equiv F(r, t)_{1/2}Y_{jm}(\theta, \phi) \end{aligned} \tag{5.15}$$

where $j \geq 1/2$, $-j \leq m \leq j$, and we have used the fact that the components u_- and v_- have spin-weight $-1/2$ and the components u_+ and v_+ have spin-weight 1/2.

Introducing the functions (5.15) in (5.14) and using the relations

$$\eth\,_{-1/2}Y_{jm}(\theta, \phi) = \left(j + \frac{1}{2}\right)\,_{-1/2}Y_{jm}(\theta, \phi)$$

$$\bar{\eth}\,_{1/2}Y_{jm}(\theta, \phi) = -\left(j + \frac{1}{2}\right)\,_{-1/2}Y_{jm}(\theta, \phi)$$

we obtain the following system of partial differential equations:

$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial g}{\partial t} - r\left(1 - \frac{t}{R}\right)\frac{\partial f}{\partial t} + \left(t + R + \frac{r^2}{R}\right)\frac{\partial f}{\partial r} + \frac{rt}{R}\frac{\partial g}{\partial r}$$
$$= \left(j + \frac{1}{2}\right)G - (t + R)\left(j + \frac{1}{2}\right)\frac{1}{r}F + (1 + k)g - (t + R)\frac{1}{r}f$$
$$R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial g}{\partial t} + r\left(1 - \frac{t}{R}\right)\frac{\partial F}{\partial t} - \left(t + R + \frac{r^2}{R}\right)\frac{\partial F}{\partial r} + \frac{rt}{R}\frac{\partial G}{\partial r}$$
$$= \left(j + \frac{1}{2}\right)g + (t + R)\left(j + \frac{1}{2}\right)\frac{1}{r}f + (1 + k)G + (t + R)\frac{1}{r}F$$
$$r\left(1 + \frac{t}{R}\right)\frac{\partial g}{\partial t} + R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial f}{\partial t} - \left(t - R - \frac{r^2}{R}\right)\frac{\partial g}{\partial r} + \frac{rt}{R}\frac{\partial f}{\partial r} \tag{5.16}$$
$$= (t - R)\left(j + \frac{1}{2}\right)\frac{1}{r}G - \left(j + \frac{1}{2}\right)F + (t - R)\frac{1}{r}g - (1 + k)f$$
$$r\left(1 + \frac{t}{R}\right)\frac{\partial G}{\partial t} - R\left(1 + \frac{t^2}{R^2}\right)\frac{\partial F}{\partial t} - \left(t - R - \frac{r^2}{R}\right)\frac{\partial G}{\partial r} - \frac{rt}{R}\frac{\partial F}{\partial r}$$
$$= (t - R)\left(j + \frac{1}{2}\right)\frac{1}{r}g + \left(j + \frac{1}{2}\right)f + (t - R)\frac{1}{r}G + (1 + k)F$$

We note that solving this system implies a completely explicit solution for the Dirac wave equation in the de Sitter universe.

6. A PARTICULAR CASE

In this section we solve the system (5.16) in the limit $R \to \infty$. We note that in this case that the system becomes

$$\frac{\partial A}{\partial t} - \frac{\partial B}{\partial r} = \frac{1}{r}\left(\frac{1}{2} - j\right)B + \frac{m}{i\hbar}A$$
$$\frac{\partial B}{\partial t} - \frac{\partial A}{\partial r} = \frac{1}{r}\left(\frac{3}{2} + j\right)A - \frac{m}{i\hbar}B \tag{6.1a}$$
$$\frac{\partial C}{\partial t} - \frac{\partial D}{\partial r} = \frac{1}{r}\left(\frac{3}{2} + j\right)D + \frac{m}{i\hbar}C$$
$$\frac{\partial D}{\partial t} - \frac{\partial C}{\partial r} = \frac{1}{r}\left(\frac{1}{2} - j\right)C - \frac{m}{i\hbar}D \tag{6.1b}$$

where we have put

$$A = g + G; \qquad B = F - f; \qquad C = G - g; \qquad D = F + f \quad (6.2)$$

Then the system (6.1a) is equivalent to

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2 A}{\partial r^2} + \frac{2}{r}\frac{\partial A}{\partial r} - \left(j + \frac{1}{2}\right)\left(j + \frac{3}{2}\right)\frac{1}{r^2}A - \frac{m^2}{\hbar^2}A \quad (6.3)$$

We put $A(t, r) = A_1(t)A_2(r)$ and obtain the ordinary differential equations

$$\frac{d^2 A_1}{dt^2} + \lambda^2 A_1 = 0 \quad (6.4a)$$

$$\frac{d^2 A_2}{dr^2} + \frac{2}{r}\frac{dA_2}{dr} + k_0^2 A_2 - \left(j + \frac{1}{2}\right)\left(j + \frac{3}{2}\right)\frac{1}{r^2}A_2 = 0 \quad (6.4b)$$

where λ is a constant and $k_0^2 = \lambda^2 - m^2/\hbar^2$.

The first of the above differential equations has the usual solutions given by

$$A_1(t) = \exp\{\pm i\lambda t\} \quad (6.5)$$

and a regular solution at the origin for the differential equation (6.4b) is a multiple of the spherical Bessel function $J_{j+1/2}(k_0 r)$

$$A_2(r) = \alpha J_{j+1/2}(k_0 r) \quad (6.6)$$

where α is a constant.

Then, using (6.5) and (6.6), we get

$$A(t, r) = \alpha \exp\{\pm i\lambda t\}\, J_{j+1/2}(k_0 r) \quad (6.7)$$

Introducing the above function in the second equation of the system (6.1a), we obtain

$$\frac{\partial B}{\partial t} + \frac{m}{i\hbar}B = \alpha \exp\{\pm i\lambda t\}\left[k_0 J_{j-1/2}(k_0 r) + \frac{1}{r}J_{j+1/2}(k_0 r)\right] \quad (6.8)$$

To solve this partial differential equation we use the method of separation of variables. Introducing the function $B(t, r)$ as a product $B(t, r) = B_1(t)B_2(r)$, we obtain

$$\frac{dB_1}{dt} + \frac{m}{i\hbar}B_1 - \lambda \exp\{\pm i\lambda t\} = 0 \quad (6.9)$$

$$B_2(r) = \frac{d}{\lambda}\left[k_0 J_{j-1/2}(k_0 r) + \frac{1}{r}J_{j+1/2}(k_0 r)\right] \quad (6.10)$$

where d is a constant.

Then from the above equations we obtain a regular solution at the origin for $B(t, r)$ as follows:

$$B(t, r) = \frac{ck_0}{m/i\hbar \pm i\lambda} \exp\{\pm i\lambda t\}\, J_{j-1/2}(k_0 r) \tag{6.11}$$

Analogously, we solve the system (6.1b) and we obtain the solutions

$$C(t, r) = a \exp\{\pm i\gamma t\}\, J_{j-1/2}(k_0 r) \tag{6.12}$$

$$D(t, r) = \frac{ak_0}{m/i\hbar \pm i\gamma} \exp\{\pm i\gamma t\}\, J_{j+1/2}(k_0 r) \tag{6.13}$$

where a and γ are constant.

Finally, using (5.15), we obtain the solution of the Dirac equation in the flat space,

$$\begin{bmatrix} u_- \\ u_+ \\ v_- \\ v_+ \end{bmatrix} = \begin{bmatrix} A(t, r)X^m_{j+1/2} \\ B(t, r)X^m_{-j-1/2} \end{bmatrix} + \begin{bmatrix} C(t, r)X^m_{-j-1/2} \\ D(t, r)X^m_{j+1/2} \end{bmatrix} \tag{6.14}$$

where the functions A, B, C, and D are given by (6.7), (6.11), (6.12), and (6.13) respectively and the $X^m_{\pm j\pm 1/2}$ are given by

$$X^m_{j+1/2} = \frac{1}{2}\begin{bmatrix} {}_{-1/2}Y_{jm} \\ {}_{1/2}Y_{jm} \end{bmatrix}; \qquad X^m_{-j-1/2} = \frac{1}{2}\begin{bmatrix} -{}_{-1/2}Y_{jm} \\ {}_{1/2}Y_{jm} \end{bmatrix} \tag{6.15}$$

Another particular case (stationary case) can be discussed in terms of the $E^l_n(\rho)$ and $G^l_n(\rho)$ polynomials (Gomes and Capelas de Oliveira, 1996).

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